Ministry of Higher Education and Scientific Research

Diyala University \ College of Science

Mathematic Department

Complex Analysis

Fourth Year

By

Assist. Lec. Asmaa Khwam Abdul-Rahman

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403MACA Complex Analysis units 8

Theoretical 4hr/week

Tutorial 4hr/week

Practical - hr/week

1- Complex Numbers

complex number definition, properties Geometric representation, Root of complex number, field of complex number as metric field .

(12hrs)

2-Regions in The Complex Plane

Open set, Close set in a complex plan, connectedness, Region, smooth Carve.

(4hrs)

3-Analytic Function

Function of a complex Variable, Limits, Continuity, Derivatives, Cauchy- Riemann Equations, Analytic Function, Harmonic Functions.

(16hrs)

4- Elementary Functions

Exponential Function, Trigonometric Function, Logarithmic Function, Hyperbolic Functions.

(8hrs)

5-Serise

Convergence of Sequence, Convergence of Series, Pour Series, Convergence Pour Series, Taylor Series, Laure Series.

(12 hrs)

6-Integrals

Definition Integrals of Function, Contour Integrals , Cauchy_Goursat Theorem, Cauchy_Integral Formula, Liouville's Theorem and the Fundamental Theorem of Algebra.

(16hrs)

7-Residues and Poles

Residues, Cauchy's Residue Theorem, Using a Single Residue, Singular Points, Zeros of Analytic Functions.

(12hrs)

8-Applications of Residues

Evaluation of Improper Integrals, Jordan's Lemma, Definit Integrals involving Sines and Cosines, Argument Principle , Rouch's Theorem.

(12hrs)

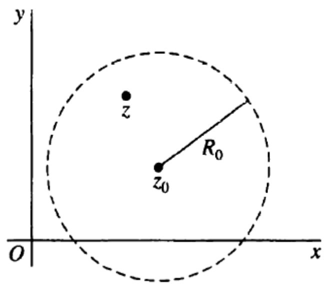
(Taylor series)

Definition:-(Taylor series)

Taylor series is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point .

Theorem:- (Taylor series)

Suppose that a function F is analytic throughout a disk $|z - z_0| < R_0$, centered at z_0 and with radius R_0 ,then $f(z)$ has the power series representation



$$f(z) = \sum_{n=0}^{\infty} an(z - z_0)^n \quad |z - z_0| < R_0 . \ . \ . (1,1)$$

Where $a_n = \frac{f^n(z_0)}{n!}$ ... $(n = 0,1,2, \ldots\ldots)$

That is series (1,1) converges to $f(z)$ when z lies in the open disk,

Series (1,1) can of course be written :

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + . \ . \ . (|z - z_0| < R_0) \quad (1,2)$$

Example:-

Since the function $f(z) = e^z$ is entire it has a maclaurin series representation which is valid for all z . Her $f^n(z) = e^X$ and because $f^n(0) = 1$, it follows that

$$e^X = \sum_{n=0}^{\infty} \left(\frac{z^n}{n!}\right) \quad (|z| < \infty) \qquad . \ . \ . \ (1,3)$$

Not that if $z = x + i0$, expansion because

$$e^x = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right) \quad (-\infty < x < \infty)$$

The entire function $z^2 e^{3z}$ also has amaclaurin series expansion. The simplest way to obtain it is replace z by $3z$ on each side of equation(2)and then multiply through the resulting by z^2

$$z^2 e^{3z} = \sum_{n=0}^{\infty} \left(\frac{3^n}{n!}\, z^{n+2}\right) \quad (-\infty < x < \infty)$$

Finally ,if we replace n by n-2 here have

$$z^2 e^{3z} = \sum_{n=2}^{\infty} \left(\frac{3^{n-2}}{n(n-2)!}\, z^n\right) \quad (|z| < \infty) \ldots$$

Find the Taylor series around $z = 0$ $f(Z) = \frac{1}{1-z}$?

Sol:

$$f(Z) = \frac{1}{1-z} \rightarrow f(Z) = \frac{1}{1-0} = 1$$

$$f'(z_0) = \frac{1}{(1-z)^2} \rightarrow f'(0) = 1$$

$$f''(z_0) = \frac{-2}{(1-z)^3} \rightarrow f''(0) = -2$$

$$f'''(z_0) = \frac{6}{(1-z)^4} \rightarrow f'''(0) = 6$$

$$f^{n}(z_0) = \frac{n!}{(1-z)^{n+1}} \rightarrow f^{n}(0) = \frac{n!}{(1-0)^{n+1}} = n!$$

$$an = \frac{f^{n}(z_0)}{n!} = \frac{n!}{n!} = 1$$

$$f(Z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} a_n (z - z_0)^n \rightarrow f(0) = \sum_{n=0}^{\infty} z^n$$

Remark: Taylor series around some function

$$f(z) = e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$f(z) = \frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots = \sum_{n=0}^{\infty} z^n$$

$$f(z) = \frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n z^n$$

$$f(z) = \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - . . . = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{2n+1}$$

$$f(z) = \cos z = 1 - \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{2n}$$

(Laurent series)

Definition (Laurent series)

Laurent series of a complex function $f(z)$ is a representation of that function as power series which includes of positive and negative degree.

(Remark)

It may be used to express complex functions in cases where a Taylor series expansion cannot be applied.

(Laurent Theorem)

Suppose that a function f is analytic throughout an annular domain R1<|z-z0|<R2 ,centered at z0 ,and let C denote any positively oriented simple closed contour a round z0 has the series representation ;

$$F(z) = \sum_{n=0}^{\infty} a_1 (z - z_0)^n + \sum_{n=0}^{\infty} \frac{bn}{(z - z_0)^n} \quad (R_1 < |z - z0| < R_2) \; . \; . \; . (1,4)$$

$$an = \frac{1}{2\pi i} \int_c \frac{f(z)dz}{(z - z_0)^{n+1}} \quad . \; . \; . \quad n = (0,1,2, \dots)$$

$$bn = \frac{1}{2\pi i} \int_c \frac{f(z)dz}{(z - z_0)^{-n+1}} \quad . \; . \; . \quad n = (0,1,2, \dots)$$

singular point :

Definition (singular point)

A point z_0 is ,called singular point of a function $f(z)$ iff F is not analytic at $z_0$but is analytic same point in every neighborhood of z_0.

(Isolated singular point)

The point z_0 is an isolated singular point of $f(z)$if $f(z)$ is not analytic at z_0 but analytic in a deleted neighborhood .

Example:-

The $function\ \frac{z+1}{z^2(z^2+1)}$ has the three isolated singular points z=0 and $z = \pm i$

Example:-the function $f(z) = \frac{1}{z}$ has a singular point at $z_0 = 0$

Example:-the function $f(z) = \log z$ has not isolated singular point

Definition (pole of order n)

$function$ whose Laurent expansion about singular point z_0 has a principal part in which the most negative power of $(z - z_0)$ is $-n$,is said to have a pole of order n at z_0 .

Example :-

The $function\ f(z) = \frac{1}{z(z-1)^2}$ has singular point at z_0 and $z = 1$ the Laurent expansion about these point is ;

$f(z) = z^{-1} + 2 + 3z + 4z^2 + \cdots \quad 0 < |z| < 1$ and

$f(z) = (z-1)^2 - (z-1)^{-1} - (z-1) + (z-1)^2 + \cdots \quad 0 < |z-1| < 1$

The first series reveals that $f(z)$ has a pole of order 1at $z = 0$,while the second shows a pole of order 2 at $z = 1$

Definition(essential singularity)

$function$ whose Laurent expansion about isolated singular point z_0 contains an infinite number of non zero terms in the principle part, is said to have an isolated essential singular at. z_0

Example:- $f(Z) = e^{\frac{1}{z}}$

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{(\frac{1}{z})^{n}}{n!}$$

$$= 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots$$

$z_0 = 0$ is called essential singular point

Definition(a removable singular point)

The point z_0 is removable singular point of the function $f(z)$ if the principle part is zero in Laurent series expansion of a function $f(z)$then $z = z_0$ is called removable singular point .

Example:-The $function\ f(z) = \frac{e^z - 1}{1}$

$$= \frac{1}{z}(e^z - 1)$$

$$= \frac{1}{z}(\sum_{n=0}^{\infty} \frac{z^n}{n!})$$

$$= \frac{1}{z}\left[1 + \frac{z}{1!} + \frac{z^2}{2!} + ... - 1\right]$$

$$= \left[1 + \frac{z}{2!} + \frac{z^2}{3!} + ...\right]$$

∴z=0 removal singularity

definition:-(Residue)

Let $f(z)$ analytic on a simple closed contour c and all point interior to c except for the point z_0 .then the residue of f(z) at z_0.written Res[f(z), z_0], is defined by:

$$Res[f(z), z_0] = \frac{1}{2\pi i}\int_c f(z)dz \quad .... (2,1)$$

Methods of finding Residue:

1- Laurent series if it is easy to write down the Laurent residue is just the coefficient $a-1$ of the term $1/(z-1)$

Note: here ,be sure you have the expansion about $z = a$; the series you have memorized for e^z, since etc,

Are expansion about z= 0 and so can be used only for finding residue at the origin.

Example

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

In analytic at z=0 ; the residue of e^z at z=0 is 0

Example :-

$$\frac{e^z}{z^3} = \frac{1}{z^3}\left\{1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right\}$$

$$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!\,z} + \frac{1}{3!} + \cdots$$

Has a pole of order 3 at z=0 ; the residue of $\frac{e^z}{z^3}$ at z=0 is $\frac{1}{2!}$

2-Residue at simple pole :

a. If $f(z)$ has simple pole at $z = a$, then the residue of $f(z)$ at $z = a$ is

$$Res\, f(a) \equiv \lim_{z\to a}(z-a)f(z)$$

Example:- find R(0)for $f(z) = \frac{cosz}{z}$:

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to a}(z-0)\frac{cosz}{z} = \lim_{z\to a} cosz = 1$$

b- Residue a pole of order n

If $f(z)$ has a pole of order n at $z = a$, then

$$\text{Res}_{z=a} f(z) = \lim_{z\to a} \frac{1}{n-1}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^{n}f(z)]\right\}$$

Example:- find the residue of $f(z) = \frac{zsinz}{(z-\pi)^3}\ at\ z = \pi$

$$\operatorname*{Res}_{z=\pi} f(z) = \lim_{z\to\pi} \frac{1}{2!}\left\{\frac{d^2}{dz^2}\left[(z-\pi)^3.\frac{zsinz}{(z-1)^3}\right]\right\}$$

$$= \lim_{z\to\pi} \frac{1}{2!}.\frac{d^2}{dz^2}(zsinz)$$

$$= \lim_{z\to\pi} = \frac{1}{2!}.\frac{d}{dz}\{zcosz + sinz\}$$

$$= \lim_{z\to\pi} \frac{1}{2!}\{z(-\sin z)$$

$$= \frac{1}{2}\{-\pi sin\pi + 2cos\pi\} = \frac{1}{2}\{-\pi.0 + 2.(-1)\} = -1$$

3- If f(z) is of the form f(z)=$\frac{g(z)}{h(z)}$ $where\ h(a) = 0\ but\ g(a) \neq 0\ then$

$$Res(z=a) = R(a) = g(a)\backslash h`(a) \quad (h`(a) \neq 0)$$

4- Residue at a pole $Z = a$ of any order (simple or of order n)

$$\operatorname*{Res}_{z=0} f(z) = R(a) = coeficient\ of\ \frac{1}{t}$$

Rule : put $z = 0$ at in the function $f(z)$, expand it in powers of t. coefficient of $\frac{1}{t}$ is the residue of f(z) at $z = a$.

Example :-find the residue of $\frac{z^3}{(z-1)^4(z-2)^1(z-3)^1}$ at z=1 here z=1 is a pole of order 4?

Putting $z = 1 + t$ or $t = z - 1$,then $f(z)$ becomes:

F(1+t)=$\frac{(1+t)^3}{t^4(t-1)(t-2)} = \frac{1}{t^4}\frac{t^3 3t^2 3t^1+1}{-(1-t).(-2x1-\frac{t}{2})}$

=$\frac{1}{t^4}(t^3 + 3t^2 + 3t + 1).\frac{1}{2}(1-t)^{-1}(1-\frac{t}{2})^{-1}$

$$= \frac{1}{2}\left(\frac{1}{t} + \frac{3}{t^2} + \frac{3}{t^3} + \frac{3}{t^4}\right) * (1 + t + t^2 + t^3 + \cdots) * \left(1 + \frac{t}{2} + \frac{t^2}{4} + \frac{t^3}{3} + \cdots\right)$$

=$\frac{1}{2}\left(\frac{1}{t} + \frac{3}{t^2} + \frac{3}{t^3} + \frac{3}{t^4}\right) * (1 + \frac{3}{2}t + \frac{7}{4}t^2 + \frac{15}{8}t^3) * \left(1 + \frac{t}{2} + \frac{t^2}{4} + \frac{t^3}{8} + \cdots\right)$

$=\frac{8+36+42+15}{16t}$

$=\frac{101}{16t}$

$$\therefore the\ coefficient\ of\ \frac{1}{2}=\frac{101}{16}, i.e. residue=\frac{101}{16}$$

5-**Residue at infinity :-**

In general ,the residue at infinity is given by :

$Res(f(z),\infty)=-Res\left(\frac{1}{z^2}f\left(\frac{1}{z}\right),0\right)$,If the following condition is met $\lim_{|z|\to\infty} f(z)=0$,then the residue at infinity can be computed using the following formula $Res(f,\infty)=-\lim_{|z|\to\infty} z.f(z)$, If instead $\lim_{|z|\to\infty} f(z)=c\neq 0$,then the Residue at infinity is $Res(f,\infty)=-\lim_{|z|\to\infty} z^2.f^{`}(z)$

Example :-

$$f(z)=\oint\frac{e^z}{z^5}dz$$

$$=\oint\frac{1}{z^5}(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots dz$$

$$=\oint(\frac{1}{z^5}+\frac{z}{z^5}+\frac{z^2}{2!\,z^5}+\frac{z^3}{3!\,z^5}+\cdots)$$

$$=\oint\frac{1}{z^5}+\frac{1}{z^4}+\frac{1}{2!\,z^3}+\frac{1}{3!\,z^2}+\cdots$$

Since the series converges uniformly on the support of the integration path we are allowed to exchange integration and summation the series of the path integrals then collapses a much simple

$$\oint_c \frac{1}{z^n}dz=0$$

$$n\in z\ for\ n\neq 1,\oint\frac{1}{z^n}$$

$$\oint\frac{1}{4!\,z}dz=\frac{1}{4!}\oint\frac{1}{z}dz=\frac{1}{4!}(2\pi i)=\frac{\pi i}{12}$$

The value $\frac{1}{4!}$ is the residue of $\frac{e^z}{z^5}$ $at\ z=0$

(Residue theorem)

Let f be analytic function on and inside the simple closed contour expect finite number of singular points

$$z_1, z_2, \dots, inside\ c\ if:$$

b_1=Res(f,z_1) , b_2=Res(f,z_2),…,b_n=(f,z_3)

There=$2\pi i \sum_{k=1}^{n} Res(f)$

$\int f(z)dz = 2\pi i(b_1+b_2+\dots+b_n)$

Proof:

Let $S = C \cup -C_1 \cup -C_2 \cup, \dots, \cup -C_n$

By Cauchy contour theorem

$$0=\int_s f(z)dz = \int_{c\cup -c_1 \cup - \dots \cup -c_n} f(z)dz$$

$$0= \int_c f(z)dz + \int_{-c_1} f_1(z)dz \quad . \; . \; . + \int_{-cn} f(z)dz$$

$$0 = \int_c f(z)dz - \int_{c_1} f_1(z)dz \;\; . \; . \; . \;\; - \int_{cn} f(z)dz$$

$$\int_c f(z)dz = \int_{c_1} f(z)dz + \int_{c_2} f_1(z)dz + \cdots + \int_{cn} f(z)dz$$

$$= 2\pi i\, Res(f, z_1) + 2\pi i\, Res(f, z_2) + \cdots + 2\pi i\, Res(f, z_n)$$

$$=2\pi i\, b_1 + 2\pi i\, b_2 + \cdots + 2\pi i\, b_n$$

$$=2\pi i\, \left(\sum_{i=1}^{n} bj\right)$$

theorem(generalize residue theorem)

Suppose that function f(z) is analytic in a closed region D bounded by the closed path C, except for a finite number of singular points,$z_1, z_2, \dots, z_n$,lying inside D and a finite number of simple poles , $z_1, z_2, \dots, z_n$, lying an C at point where C is smooth then:

$$p.v. \int_c f(z)dz = 2\pi i \sum_{k=1\ z=zk}^{n} Resf(z) + \pi i \sum_{k=1\ z=zn}^{1} Resf(z)$$

Proof:

We bypass each singular z_k by a circular are $y_k$of radius $\mathcal{S}$ and center at z_k,lying in D .we choose $\mathcal{S}$ .so small that the whole are $y_k$lies in the region of analytic of f(z) .then f(z) is analytic on the closed path which consists of the arc γ_k and the remaining part ,$\tilde{c}$ of c therefore by the residue theorem :

$$\int_{c} f(z)dz + \sum_{k=1}^{n} \int_{\gamma_k} f(z)dz = 2\pi i \sum_{k=1}^{1} \underset{z=zk}{Resf(z)}$$

Expanding f(z) in a Laurent series in a neighborhood of the simple pole z_k we obtain :

$$F(z) = \frac{c-1}{z-z_k}\mathrm{dz} + \sum_{\mathrm{n}}^{\infty} \mathrm{c_n}(\mathrm{z_n} - \tilde{\mathrm{z}}_{\mathrm{k}})^{\mathrm{n}}$$

Then:

$$\int_{\gamma_k} f(z)dz = \int_{\gamma_k} \frac{c-1}{z-\tilde{\mathrm{z}}_{\mathrm{k}}} dz + \sum_{n=0}^{\infty} c_n(z_n - \tilde{c}_k)^{\mathrm{n}}\, dz$$

On the arc γk we have $z = \tilde{z}k + se^{io}$ $, \propto_k \leq 0 \leq B_k , where \propto_k$

Is the angle between the secant joining the point A_k and $\tilde{z}k$ and the tangent to $\tilde{c}$ at $\tilde{z}k$ and B_k and $\tilde{z}k$ and the same tangent (see the magnification of arc γk in fig become .

$$\int_{\gamma_k} f(z)dz = c-1 \int_{\propto k}^{Bk} \frac{se^{io} i d\theta}{se^{io}} + \sum_{n=1}^{\infty} cn \int_{\propto k}^{Bk} \left(se^{io}\right)^{\mathrm{n}} se^{io} i d\theta$$

In the limit , as $s \to 0$,we have $\propto_k \to \pi , B_k \to 0 , become$:

$$\lim_{s\to 0} \int_{\gamma_k} f(z)dz = ic - 1 \int_{\pi}^{o} d\theta$$

$$= -\pi ic - 1$$

$$= -\pi\, Res_{z=\tilde{z}\,k} f(z)$$

Chance ,taking the limit as $\delta n \to 0$ we obtain In the case the points $\tilde{z}k$ are pales of any odd order $(\tilde{z}k)$and the principal part of the Laurent series contains only odd power of $z - \tilde{z}k$

$f(z) = \sum_{p=0}^{\delta} \frac{c-(2p+1)}{(z-\tilde{z}k)^{2p+1}} + \sum_{n=0}^{\infty} cn(z-\tilde{z}k)^{\mathrm{n}} , where\ c - (2p+1) \neq 0$

Ended, integrating each of the terms in the principal along the arc yk from $\theta = \pi\ to\ \theta = 0$ we obtain, as in the transition, that the term containing c-1 is the only nonzero from this term is:

$$\int_{\gamma_k} \frac{dz}{(z-\tilde{z}k)^{2p+1}} = \int_{\pi}^{o} \frac{e^{io} i d\theta}{(e^{io})^{2p+1}} = i \int_{\pi}^{o} e^{-2pi\theta} d\theta$$

$$= \int_0^{-\pi i} \begin{matrix} if\ p = 0 \\ ifp = 1,2, \dots, \delta \end{matrix}$$

Note the simple pole of the integrands located on the path accrue in diffraction problems

<u>Example:</u> Evaluate the following integral counterclockwise :

$$I_6 = p.v. \int_{121=1} \frac{sinz}{(z^2-1)(z^2+1)} dz$$

<u>Solution :-</u>

the four singular points, $z = \pm 1$ and $z = \pm i$, of the integrand are simple poles. Moreover, all the singularities are located on the circle $|z| = 1$

$$I_6 = \pi i (Res_{z=1} + Res_{z=-1} + Res_{z=i} + Res_{z=-i}) \left[\frac{sinz}{(z^2-1)(z^2+1)}\right]$$

$$\pi i \left[\frac{sinz}{2z(z^2+1)}\right] z = i + \frac{sinz}{2z(z^2-1)} [z=-i]$$

$$= \pi i \left[\frac{sin1}{4} + \frac{sin1}{4} + \frac{sin1}{2i(-2)} + \frac{\sin(-1)}{2(-i)(-2)}\right]$$

$$= \frac{\pi i}{2} (sin1 - \sin hi)$$